2 A small investment problem

Example 2. Suppose you have \$5,000 to invest. Over the next 3 years, you want to double your money. At the beginning of each of the next 3 years, you have an opportunity to invest in one of two investments: A or B. Both investments have uncertain profits. For an investment of \$5,000, the profits are as follows:

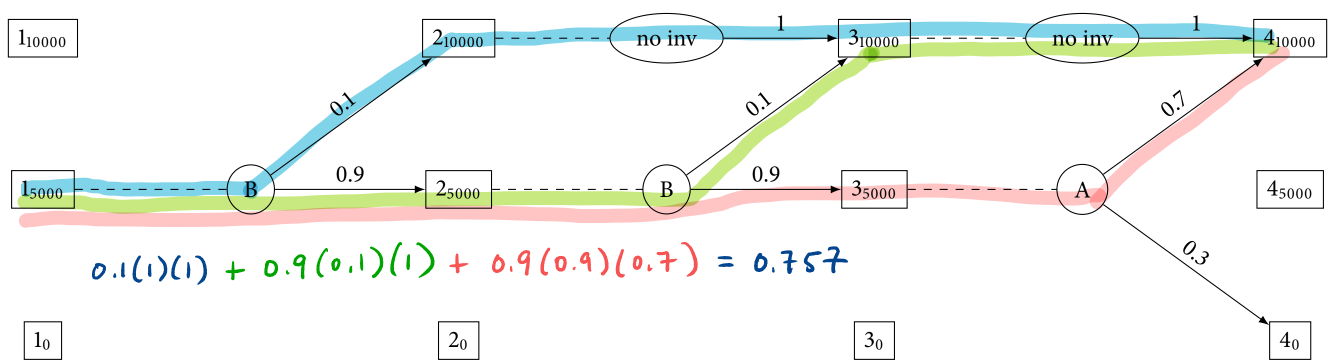
Investment	Profit (\$)	Probability
A	−5, 000	0.3
	5,000	0.7
B	0	0.9
	5,000	0.1

You are allowed to make at most one investment each year, and can invest only \$5,000 each time. Any additional money accumulated is left idle. Once you've accumulated \$10,000, you stop investing.

Formulate a stochastic dynamic program to find an investment policy that maximizes the probability you will have \$10,000 after 3 years.

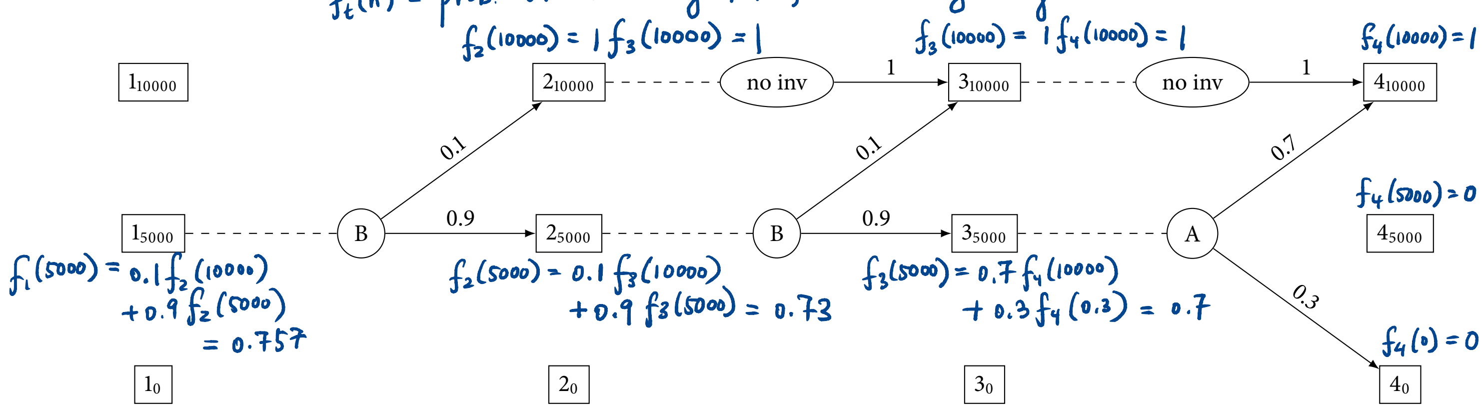
2.1 Warm up

Consider the following investment policy. What is the probability of having at least \$10,000?



$0.1(1)(1) + 0.9(0.1)(1) + 0.9(0.9)(0.7) = 0.757$

$f_t(n)$ = prob. of finishing w/ \$10,000 starting at year t w/ n dollars



2.2 Formulating the stochastic dynamic program

- Stages:

Stage t represents the beginning of year t $(t = 1, 2, 3)$, or
the end of the decision-making process $(t = 4)$

- States:

State n represents having n dollars $(n = 0, 5000, 10000)$

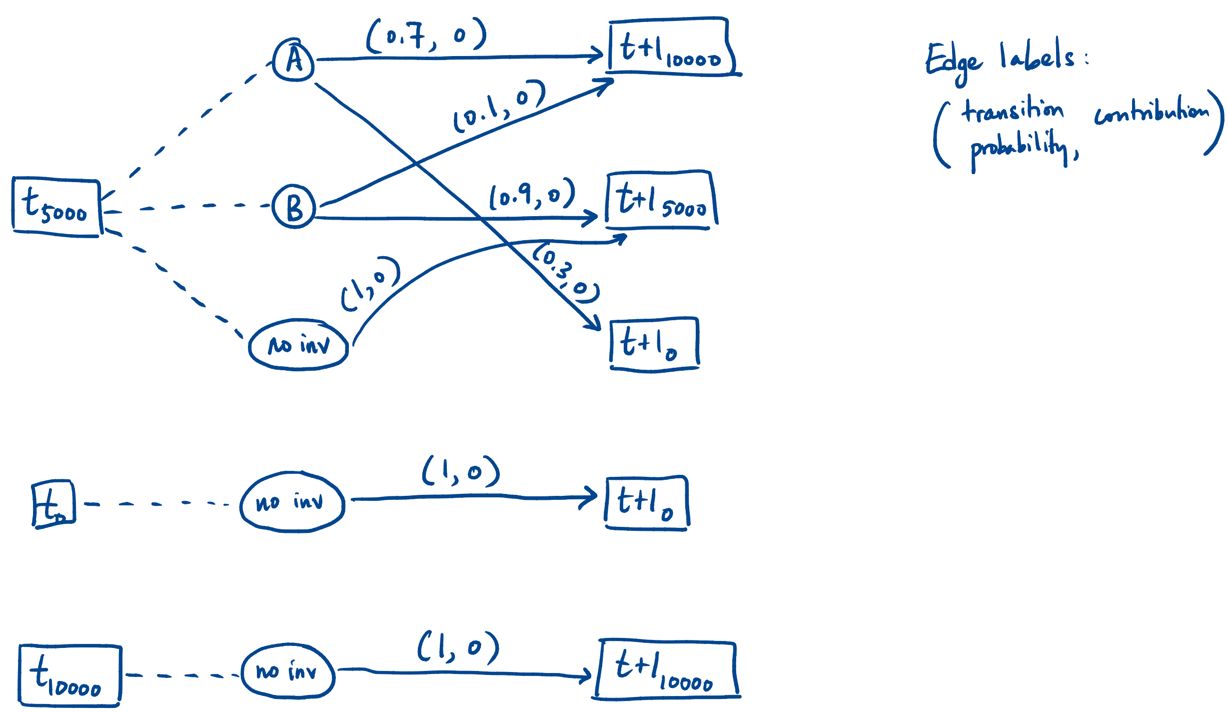
- Allowable decisions x_t at stage t and state n:

$t = 1, 2, 3$: Let x_t = investment to make in year t

x_t must satisfy:
$$x_t \in \begin{cases} \{A, B, \text{no investment}\} & \text{if } n = 5000 \\ \{\text{no investment}\} & \text{if } n = 0, 10000 \end{cases}$$

$t = 4$: No decisions

- Sketch of basic structure – transition probabilities and contributions:



- In words, the value-to-go $f_t(n)$ at stage t and state n is:

$f_t(n)$ = maximum probability of finishing w/\$10,000, starting at year t with n dollars

for $t = 1, \dots, 4$ and $n = 0, 5000, 10000$

- Value-to-go recursion

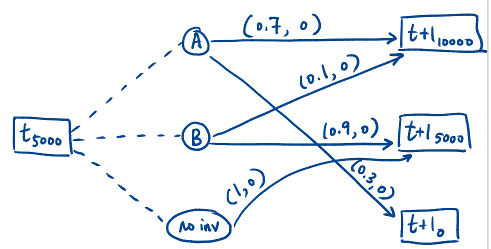
$$f_t(n) = \min_{x_t \text{ allowable}} / \max \left\{ \sum_{m \text{ state}} p(m \mid n, t, x_t) \left[c(m \mid n, t, x_t) + f_{t+1}(m) \right] \right\} \quad \text{for stages } t \text{ and states } n$$

$$f_t(5000) = \max \Big\{ \underbrace{0.7 f_{t+1}(10000) + 0.3 f_{t+1}(0)}_{x_t = A},\ \underbrace{0.1 f_{t+1}(10000) + 0.9 f_{t+1}(5000)}_{x_t = B},\ \underbrace{1 f_{t+1}(5000)}_{x_t = \text{no inv.}} \Big\}$$

max. prob. of finishing w/\$10,000 starting at year t w/\$5000.

$$f_t(10000) = \max \{ 1 \cdot f_{t+1}(10000) \} = f_{t+1}(10000)$$

$$f_t(0) = \max \{ 1 \cdot f_{t+1}(0) \} = f_{t+1}(0)$$



- Boundary conditions:

$f_4(10000) = 1$ $\qquad f_4(5000) = 0$ $\qquad f_4(0) = 0$

- Desired value-to-go function value: $f_1(5000)$

2.3 Interpreting the value-to-go function

- Solving the recursion on $f_t(n)$, we obtain:

t	n	$f_t(n)$	x_t^*
1	0	0	no investment
1	5000	0.757	B
1	10000	1	no investment
2	0	0	no investment
2	5000	0.73	B
2	10000	1	no investment
3	0	0	no investment
3	5000	0.7	A
3	10000	1	no investment

t	n
2	10000
2	5000

- Based on this, what should your investment policy be?

Year 1: Invest in B

Year 2: If $n = 5000$, invest in B
If $n = 10000$, no investment
$\Rightarrow$ we're going to end up in states 5000 or 10000 in year 3

Year 3: If $n = 5000$, invest in A
If $n = 10000$, no investment

- What is your probability of having \$10,000?

$f_1(5000) = 0.757$